Cost Minimization

1 Problem Set Up

Cost minimization problems are the analog of utility maximization problems. The good news is, you already know all of the mathematical techniques you need to solve these problems, the only aspect that is new mathematically is how to set up the firm's cost minimization problem.

A natural question is, what does this exactly mean? A firm produces a good, call it q. How they produce it is through their production function, $f(L, K)$, where L is labor and K is capital, and f is just the function that maps the inputs into the number of goods produced. We can express the production function as

$$q = f(L, K)$$

Now, this tells us absolutely nothing about the firm's costs. We know that the firm uses two things to produce their output, so their total cost will be a function of the amount of labor and capital they use. Let w be the price paid to labor (the wage) and r be the price paid to capital (otherwise known as the rental rate), the total cost to the firm is

$$c(q) = wL^*(w, r, q) + rK^*(w, r, q)$$

Notice that total costs are a function of the amount of output produced by the firm, that is fairly straightforward. What may be less clear are the functions $L^*(w, r, q)$ and $K^*(w, r, q)$, but these are just the firm's demand functions for labor and capital! The amount of labor a firm uses will be a function of the price the firm has to pay workers (w), the price they have to pay to capital (r), and the output the want to produce (q).

Finally, we can set up the firm's cost minimization problem. Recall that a consumer maximizes their utility function subject to a budget constraint, the utility function is their objective function, and their budget constraint is the constraint they face. In a cost minimization problem, the firm's objective function is to minimize c(q) subject to their production function $q = f(L, K)$. The LaGrangian is

$$\ell = \min_{L,K} wL + rK - \lambda(f(L, K) - q)$$

We take the first order conditions with respect to L, K, and λ, this produces the following system of equations:

$$\frac{\partial \ell}{\partial L} = w - \lambda \frac{\partial f(L, K)}{\partial L} = 0 \quad (1)$$

$$\frac{\partial \ell}{\partial K} = r - \lambda \frac{\partial f(L,K)}{\partial K} = 0 \tag{2}$$

$$\frac{\partial \ell}{\partial \lambda} = f(L,K) - q = 0 \tag{3}$$

If we isolate λ in (1) and (2), we derive

$$\frac{w}{\frac{\partial f(L,K)}{\partial L}} = \lambda = \frac{r}{\frac{\partial f(L,K)}{\partial K}} \tag{4}$$

Re-arranging yields

$$\frac{\frac{\partial f(L,K)}{\partial L}}{\frac{\partial f(L,K)}{\partial K}} = \frac{w}{r} \tag{5}$$

The term $\frac{\partial f(L,K)}{\partial L}$ is the marginal product of labor and $\frac{\partial f(L,K)}{\partial K}$ is the marginal product of capital, therefore, we have arrived at the familiar condition that the optimal bundle of L and K is when the ratio of the marginal products equals the price ratio. Here, we use the term the Marginal Rate of Technical Substitution (MRTS).

2 Example

Suppose $q = f(L,K) = L^{\frac{2}{3}}K^{\frac{1}{3}}$, lets solve the firm's cost minimization problem to obtain their demand functions $L^*(w,r,q)$ and $K^*(w,r,q)$. The LaGrangian is

$$\ell = \min_{L,K} wL + rK - \lambda(L^{\frac{2}{3}}K^{\frac{1}{3}} - q)$$

Setting the MRTS equal to the price ratio yields:

$$\frac{\frac{2}{3}L^{-\frac{1}{3}}K^{\frac{1}{3}}}{\frac{1}{3}L^{\frac{2}{3}}K^{-\frac{2}{3}}} = \frac{w}{r} \tag{6}$$

which reduces to

$$\frac{2K}{L} = \frac{w}{r} \tag{7}$$

We can use this to solve for K in terms of L, this gives us

$$K = \frac{wL}{2r} \tag{8}$$

Now, with a consumer in a utility maximization problem, we would plug this into their budget constraint to solve for the demand functions. We do not have a budget constraint in this problem, we have an output constraint, therefore we plug (8) into $q = f(L,K) = L^{\frac{2}{3}}K^{\frac{1}{3}}$.

$$q = L^{\frac{2}{3}}K^{\frac{1}{3}} \tag{9}$$

$$q = L^{\frac{2}{3}}(\frac{wL}{2r})^{\frac{1}{3}} \tag{10}$$

$$q = L^{\frac{2}{3}} L^{\frac{1}{3}} (\frac{w}{2r})^{\frac{1}{3}} \tag{11}$$

$$q = L(\frac{w}{2r})^{\frac{1}{3}} \tag{12}$$

then our demand function for labor is:

$$L^*(w, r, q) = q(\frac{w}{2r})^{-\frac{1}{3}} \tag{13}$$

To find the demand for capital, we just use (8) and plug in (13), doing so will yield

$$K^*(w, r, q) = q(\frac{w}{2r})^{\frac{2}{3}} \tag{14}$$

if you do not like negative exponents, you can re-write the demand for labor as

$$L^*(w, r, q) = q(\frac{2r}{w})^{\frac{1}{3}} \tag{15}$$

A few properties become apparent, the demand for labor and capital is increasing in output (q). This is expected, to produce more stuff, the firm needs more of each input. Further, the partial derivative of labor with respect to wage is negative, indicating that as labor becomes more expensive, the firm demands less of it, another intuitive result, as one input becomes relatively more expensive, the firm will substitute away from it. The same analogy applies for capital, as r increases, the demand for capital decreases.

Finally, we can write the firm's cost function

$$c(q) = w(q(\frac{2r}{w})^{\frac{1}{3}}) + r(q(\frac{w}{2r})^{\frac{2}{3}}) \tag{16}$$