Utility Maximization

The basic problem that a consumer faces is to maximize their utility function, $u(x, y)$, subject to their budget constraint $p_x x + p_y y \leq I$. With monotonic preferences, i.e, more of each good is better, the consumer will exhaust their budget constraint $p_x x + p_y y = I$. How do we solve these types of problems?

1 Lagrangian Approach

When $u(x, y)$ generates indifference curves that are convex, we use the Lagrangian approach. Convex indifference curves indicate that the consumer likes more of each good, but there is diminishing marginal utility. In this case, the we solve

$$L(x, y) = \max_{x,y} u(x, y) - \lambda(p_x x + p_y y - I)$$

where λ is the Lagrange multiplier. To derive the optimal bundle (x^*, y^*) we take first order conditions with respect to x, y, and λ, and obtain following system of equations:

$$\frac{\partial L}{\partial x} = \frac{\partial u(x, y)}{\partial x} - \lambda p_x = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} = \frac{\partial u(x, y)}{\partial y} - \lambda p_y = 0 \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = p_x x + p_y y - I = 0 \tag{3}$$

Now, we have three equations and three unknowns. Use (1) and (2) to substitute out λ and derive a relationship between x and y. This yields

$$\frac{\frac{\partial u(x,y)}{\partial x}}{p_x} = \frac{\frac{\partial u(x,y)}{\partial y}}{p_y}$$

or

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

once you have this relationship between x and y, you utilize the budget constraint to solve for the optimal bundles.

Notice from the last equation that we can write

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

which means that at the optimal bundle, the marginal rate of substitution must equal the price ratio. This means, that if you are able to identify that we have preferences that generate convex indifference curves, you can just derive this relationship and solve for the optimal bundle using

$$MRS = MRT$$

$$p_x x + p_y y = I$$

1.1 Example: Cobb-Douglas Preferences

Let a consumer's preferences over goods x and y be given by

$$u(x, y) = x^{\frac{1}{4}} y^{\frac{3}{4}}$$

and suppose their budget constraint is

$$p_x x + p_y y = I$$

Solve for (x^*, y^*). This utility function is concave in both arguments, so we know that our optimality condition is

$$MRS = MRT$$

or

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

Computing marginal utilities yields:

$$MU_x = \frac{1}{4} x^{-\frac{3}{4}} y^{\frac{3}{4}}$$

$$MU_y = \frac{3}{4} x^{\frac{1}{4}} y^{-\frac{1}{4}}$$

and our optimality condition is

$$\frac{\frac{1}{4} x^{-\frac{3}{4}} y^{\frac{3}{4}}}{\frac{3}{4} x^{\frac{1}{4}} y^{-\frac{1}{4}}} = \frac{p_x}{p_y}$$

which reduces to something much nicer:

$$\frac{y}{3x} = \frac{p_x}{p_y}$$

Now, we can solve for y in terms of x

$$y = 3x \frac{p_x}{p_y}$$

and we can plug this into our budget constraint.

$$p_x x + p_y (3x \frac{p_x}{p_y}) = I$$

which means that

$$x^* = \frac{I}{4p_x}$$

We know that this is our optimal x because we do not see any y terms on the right hand side of the equation. Plug this back into the relationship we derived between x and y and we get

$$y^* = 3x^* \frac{p_x}{p_y}$$

$$y^* = \frac{3I}{4p_y}$$

Now, we can perform a little sanity check to see if our math is correct. Go back to the utility function, notice, that this consumer likes good y three time more than they like good x (this is by looking at the exponents). Suppose $I = p_x = p_y = 1$, then

$$x^* = \frac{1}{4},\ y^* = \frac{3}{4}$$

and we see that with equal prices, they do in fact buy three time more of good y.

2 Other Optimality Conditions

Unfortunately, we cannot always use the condition that the optimal bundle will occur when MRS = MRT. Here are some examples of when this approach does not work and how we can handle it.

2.1 Perfect Substitutes

If the consumers preferences do not exhibit convex indifference curves, then the optimal bundle may not be at an interior point which means the Lagrangian approach will not give us the utility maximizing bundle. Let us look at the case of perfect substitutes. Suppose a consumer's utility function is

$$u(x, y) = 2x + y$$

and their budget constraint

$$x + y = 10$$

If we use the Lagrangian approach, we derive the following system of equations

$$2 - \lambda = 0$$

$$1 - \lambda = 0$$

$$x + y - 10 = 0$$

Our first hint that the Lagrangian approach is not appropriate for perfect substitutes is that we cannot use a system of equations to solve for x and y, they do not appear. Additionally, we have derived the contradiction that $\lambda = 1$ and $\lambda = 2$, which is not true, therefore, the optimal bundle is not at an interior point. So what do we do?

Perfect substitutes generate linear indifference curves, and this implies that the utility maximizing bundle will ALWAYS be a corner solution (unless the utility weights are equal and the price of each good is the same, then the consumer is indifferent between any point on the budget line). To find the optimal bundle, we use the bang per buck approach, that is we compare $\frac{MU_x}{p_x}$ and $\frac{MU_y}{p_y}$. In this case, we have

$$\frac{MU_x}{p_x} = 2 > 1 = \frac{MU_y}{p_y}$$

so we spend all of our income on good x and the optimal bundle is

$$(x^*, y^*) = (10, 0)$$

Therefore, in the case with perfect substitutes, we use the bang per buck approach, and spend all our income on the good with the higher bang per buck.

2.2 Perfect Complements

Utility functions for perfect complements have the form

$$u(x, y) = \min\{ax, by\}$$

which is not differentiable and therefore we cannot use the Lagrangian approach. This is perhaps the easiest case to solve for, as these goods need to be consumed in fixed quantities to increase utility. To solve, we set the inside equal to derive a relationship between x and y, and then use the budget constraint. Let

$$u(x, y) = \min\{3x, y\}$$

and assume the budget constraint is

$$2x + 3y = 12$$

Setting the terms in the min function equal yields

$$y = 3x$$

Plugging this into the budget constraint and solving gives us

$$(x^*, y^*) = (\frac{12}{11}, \frac{36}{11})$$

2.3 Quasi-Linear

Quasi linear utility can have the optimal bundle be a corner solution or an interior solution and therefore requires the most care when solving. Suppose:

$$u(x, y) = 2y + \sqrt{x}$$

$$x + 8y = 2$$

Setting MRS = MRT yields:

$$\frac{x^{-\frac{1}{2}}}{4} = \frac{1}{8}$$
$$x^* = 4$$

Plug x^* into the budget constraint and we get

$$y^* = -\frac{1}{4}$$

which is infeasible, we cannot consume a negative amount of a good. What do we do? Spend all of our money on the good that came out positive in our solution, which will always be the non-linear good. In our case, that is good x, and the solution is

$$(x^*, y^*) = (2, 0)$$

Suppose now the budget constraint is

$$2 = x + 2y$$

Setting MRS = MRT yields

$$\frac{x^{-\frac{1}{2}}}{4} = \frac{1}{2}$$
$$x^* = 2$$

Plug this into the budget constraint and we get

$$(x^*, y^*) = (\frac{1}{4}, \frac{7}{8})$$

and we are done.

2.4 Concave Indifference Curves

The only case without an example is the case of concave indifference curves. Let preferences be given by

$$u(x, y) = \sqrt{x^2 + y^2}$$

and the budget constraint

$$x + y = 20$$

Setting MRS = MRT yields

$$\frac{x}{y} = 1$$

therefore the Lagrangian approach yields

$$(x^L, y^L) = (10, 10)$$

with corresponding utility

$$u(10, 10) = \sqrt{200}$$

however, check the solution (20,0), this gives utility

$$u(20, 0) = \sqrt{400} > u(x^L, y^L) = \sqrt{200}$$

so the Lagrangian approach again does not yield the utility maximizing bundle.