Math Review

1 Derivative Rules

In these notes I will cover some of the basics of multivariate calculus that will be needed to solve optimization problems in economics. In Calculus I and II, we consider functions of a single variable, f(x), where x is the input, and f(.) the function. To optimize a function, typically we will be taking derivatives and setting them equal to 0, and solving for our choice variables. Here are some important derivative rules that you should commit to memory:

1. $f(x) = ax,$ a is a constant, $\frac{df}{dx} = a$
2. $f(x) = ax^b,$ a and b are a constants, , $\frac{df}{dx} = bax^{b-1}$
3. $h(x) = f(x)g(x)$ $\frac{df}{dx} = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$
4. $h(x) = \frac{f(x)}{g(x)}$ $\frac{df}{dx} = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{g(x)^2}$
5. $fh(x) = f(g(x))$ $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$
6. $f(x) = ln(x)$ $\frac{df}{dx} = \frac{1}{x}$
7. $f(x) = e^x$ $\frac{df}{dx} = e^x$
8. $f(x) = a$ where a is a constant $\frac{df}{dx} = 0$

Rule 4 is the chain rule. Here is an example. Let

$$f(x) = ln(x^2)$$

Here, our outer function is ln(x) and our inner function is x^2. To find the derivative, we take the derivative of the outer, and multiply is by the derivative of the inner. This gives us

$$\frac{1}{x^2}2x$$

$$\frac{2}{x}$$

We can use the log rule that $ln(x^a) = aln(x)$ to check our work. In this case, we only need to consider the derivative of $ln(x)$ which we know is $\frac{1}{x}$, so we are left with

$$\frac{d2ln(x)}{dx} = \frac{2}{x}$$

2 Optimization With Single Variable Functions

Now, how do we optimize single variable functions? As mentioned above, we take derivative, set equal to 0, and solve for the optimal point x. Consider the function

$$f(x) = ln(x) - 2x$$

We want to find the optimal x. Taking derivative and setting to zero gives

$$\frac{df}{dx} = \frac{1}{x} - 2 = 0$$

Solving for x

$$\frac{1}{x} = 2$$

$$x^* = \frac{1}{2}$$

which is a critical point of the function f. If this were a calculus course, your professor would probably ask you if this corresponds to a maximum or a minimum. To answer this, you look at the second derivate of f. In this case it is

$$\frac{d^2 f}{dx^2} = -\frac{1}{x^2} < 0$$

because the second derivative is less than zero, we are at a maximum.

3 Multivariate Calculus and Some Economics

Now, let us move onto multivariate calculus, which means that we consider functions of more than one variable. Typically, we are studying consumers who have preferences over two goods, x and y, where preferences are encoded in a utility function u(x,y). The derivative rules are still the same as in the single variable case, because when we take the derivative of u(x,y) with respect to x, what we are actually doing is seeing how u(x,y) changes with x. Note that it ONLY changes with x, that means, we treat y as a constant. Here is an example

$$u(x, y) = xy$$

$$\frac{\partial u}{\partial x} = y$$

Notice, that when we take the partial derivative (only taking derivative with respect to one of the inputs to the function) the variable y is treated as a constant because we are only concerned with how u(x,y) changes when we change x. If we wanted to see how u(x,y) changes with y, we compute

$$\frac{\partial u}{\partial y} = x$$

where we treat x as if it is a constant.

Some other examples are

$$u(x, y) = 2x + y$$

Now,

$$\frac{\partial u}{\partial x} = 2$$

$$\frac{\partial u}{\partial y} = 1$$

Why? First, the partial derivative of u with respect to x. Again, we are only interested in how u changes with x, holding y constant. The derivative of 2x with respect to x is just 2, easy enough. Now, the partial derivative of y with respect to x is 0. Why? When we are taking the partial derivative of u with respect to x, we are holding everything else constant, that includes y, what is the derivative of a constant? That is right, it is 0. The same applies for the partial derivative of u with respect to y. The partial derivative of y with respect to y is just 1, and since we are treating x as a constant, the partial derivative of x with respect to y is 0.

Now, let's start doing some economics by looking at a consumer's problem. The basic problem that a consumer faces is to maximize their utility function, $u(x, y)$, subject to their budget constraint $p_x x + p_y y \leq I$. With monotonic preferences, i.e, more of each good is better, the consumer will exhaust their budget constraint $p_x x + p_y y = I$. How do we solve these types of problems?

3.1 Lagrangian Approach

The Lagrangian approach just means that we are optimizing an alternative function $L(x, y)$, which takes into account our objective function, u(x,y), and our constraint, here the budget constraint.

$$L(x, y) = \max_{x,y} u(x, y) - \lambda(p_x x + p_y y - I)$$

where λ is the Lagrange multiplier. To derive the optimal bundle (x^*, y^*) we take first order conditions with respect to x, y, and λ, and obtain following system of equations:

$$\frac{\partial L}{\partial x} = \frac{\partial u(x, y)}{\partial x} - \lambda p_x = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} = \frac{\partial u(x, y)}{\partial y} - \lambda p_y = 0 \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = p_x x + p_y y - I = 0 \tag{3}$$

Now, we have three equations and three unknowns. Use (1) and (2) to substitute out λ and derive a relationship between x and y. This yields

$$\frac{\frac{\partial u(x,y)}{\partial x}}{p_x} = \frac{\frac{\partial u(x,y)}{\partial y}}{p_y}$$

or

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

once you have this relationship between x and y, you utilize the budget constraint to solve for the optimal bundles.

Notice from the last equation that we can write

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

which means that at the optimal bundle, the marginal rate of substitution must equal the price ratio. This means, that if you are able to identify that we have preferences that generate convex indifference curves, you can just derive this relationship and solve for the optimal bundle using

$$MRS = MRT$$

$$p_x x + p_y y = I$$

3.2 Example: Cobb-Douglas Preferences

Let a consumer's preferences over goods x and y be given by

$$u(x, y) = x^{\frac{1}{4}} y^{\frac{3}{4}}$$

and suppose their budget constraint is

$$p_x x + p_y y = I$$

Solve for (x^*, y^*). This utility function is concave in both arguments, so we know that our optimality condition is

$$MRS = MRT$$

or

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

Computing marginal utilities yields:

$$MU_x = \frac{1}{4} x^{-\frac{3}{4}} y^{\frac{3}{4}}$$

$$MU_y = \frac{3}{4} x^{\frac{1}{4}} y^{-\frac{1}{4}}$$

and our optimality condition is

$$\frac{\frac{1}{4}x^{-\frac{3}{4}}y^{\frac{3}{4}}}{\frac{3}{4}x^{\frac{1}{4}}y^{-\frac{1}{4}}} = \frac{p_x}{p_y}$$

which reduces to something much nicer:

$$\frac{y}{3x} = \frac{p_x}{p_y}$$

Now, we can solve for y in terms of x

$$y = 3x\frac{p_x}{p_y}$$

and we can plug this into our budget constraint.

$$p_x x + p_y(3x\frac{p_x}{p_y}) = I$$

which means that

$$x^* = \frac{I}{4p_x}$$

We know that this is our optimal x because we do not see any y terms on the right hand side of the equation. Plug this back into the relationship we derived between x and y and we get

$$y^* = 3x^*\frac{p_x}{p_y}$$

$$y^* = \frac{3I}{4p_y}$$

Now, we can perform a little sanity check to see if our math is correct. Go back to the utility function, notice, that this consumer likes good y three time more than they like good x (this is by looking at the exponents). Suppose $I = p_x = p_y = 1$, then

$$x^* = \frac{1}{4},\ y^* = \frac{3}{4}$$

and we see that with equal prices, they do in fact buy three time more of good y.